A Retrospective Temporal Integration Method for Richards' Equation

Chen Xiangdong
Key Laboratory of Water Cycle and Related Land Surface Processes,
Institute of Geographic Sciences and Natural Resources Research, Chinese Academy of Sciences
Beijing 100101, China
Email: chenxd.08b@igsnrr.ac.cn

Xia Jun
Key Laboratory of Water Cycle and Related Land Surface Processes,
Institute of Geographic Sciences and Natural Resources Research, Chinese Academy of Sciences
Beijing 100101, China
Email: xiaj@igsnrr.ac.cn

Ye Aizhong
School of Geography
Beijing Normal University
Beijing 100875, China
Email: azye@bnu.edu.cn

Zhang Yongyong
Key Laboratory of Water Cycle and Related Land Surface Processes,
Institute of Geographic Sciences and Natural Resources Research, Chinese Academy of Sciences
Beijing 100101, China

Abstract

Numerical simulation is the effective method of solving Richards' equation for whose highly-nonlinear character. Most methods of the numerical solutions improved the special discrete methods, or designed on the basis of physical conservation. Those methods can be solved when one time initial value is given. Combining the self-memory principle, the retrospective time integration method for Richards' equation is proposed. It is a new kind of time integration scheme which could include multi-time historical data, absorbs the character of stochastic method which forecast by making use of multi-time historical data before the initial time. This scheme is applied to simulate the fixed head vertical infiltration. The results of 1-order retrospective scheme are calculated with given memory coefficients. The stable region of the memory coefficients in 1-order retrospective scheme is also given. The results show that the retrospective scheme can get higher accuracy than the implicit scheme.

1. Introduction

Soil water is the connection of surface water and groundwater and an essential element in water cycle process. So study on the rule of soil moisture movement has very important significance. The Richards' equation for soil moisture movement is a partial differential equation of second order with highly non-liner. Only with simple boundary condition or simplified soil parameters, analytical solutions can be found [1]. So to deal with natural conditions, many numerical methods have been developed to solve the Richards' equation and simulate the soil water movement, like the finite difference method, the finite element method, the mixed finite element method and the finite volume method [2-8].

Gu pointed out that the difference between numerical prediction with an initial value and the empirical prediction with several past observations before the initial time. He proposed improving the numerical prediction by making use of the past data [9]. After this, multiple time mode of numerical prediction become focus of many scholars to study. Cao based on the viewpoint that atmosphere movement is an irreversible process, introducing a memory function which can recall data in the past, a concept of self-memory in atmospheric motion is proposed [10,11]. The atmospheric differential equation is generalized to a self-memory equation which includes multiple time levels. And the computation based on the self-memory equation is called self-memory principle. The existing various modes of multiple time numerical prediction can be unified into one frame of the self-memory equation. It constructed a retrospective temporal integration method [12,13]. The motivation of our study is to simulate the soil water movement with the self-memory principle.

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2. The Richards' equation

The Richards' equation is the equation most frequently used to describe the soil water movement. The one-dimensional equation is expressed as follows

$$C(h)\frac{\partial h}{\partial t}=\frac{\partial}{\partial z}\left[K(h)\frac{\partial h}{\partial z}\right]-\frac{\partial K(h)}{\partial z} \qquad (1)$$

where h is the pressure head (cm), C(h) is the specific water capacity (cm^{-1}), K(h) is the hydraulic conductivity (cm/min), z is the depth, positively oriented downward (cm), and t is the time (min).

3. The self-memory principle

Introducing a memory function into a differential equation will produce a difference-integral equation, that is, the self-memory equations. This methodology of prediction based on the self-memory equation is different form that to solve differential equations governing motion of a system as an initial value problem [10] . And the equation is derived by Cao as follows[10].

$$\frac{\partial x}{\partial t}=F(x,\lambda,t) \qquad (2)$$

where x is the variable of the system state, λ is the parameter, t is time.

Considering a set of time T=[t_{-p}, $t_{-p+1}\dots t_0$, t], where t_0 is an initial time, applying an inner product operation to Eq.(2) by introducing a memorial function β(t), we can obtain

$$\int_{t_{-p}}^{t}\beta(\tau)\frac{\partial x}{\partial\tau}d\tau=\int_{t_{-p}}^{t}\beta(\tau)F(x,\lambda,\tau)d\tau \qquad (3)$$

Supposing the variables x and β are continuous, differentiable and integrable, following calculus, making an integration by parts, applying the median theorem, and performing algebra operation. The left-side of (3) can be obtained as follows[13]

$$\int_{t_{-p}}^{t}\beta(\tau)\frac{\partial x}{\partial\tau}d\tau=\int_{t_{-p}}^{t_{-p+1}}\beta(\tau)\frac{\partial x}{\partial\tau}d\tau+$$

$$\int_{t_{-p+1}}^{t_{-p+2}}\beta(\tau)\frac{\partial x}{\partial\tau}d\tau+\dots\dots+\int_{t_0}^{t}\beta(\tau)\frac{\partial x}{\partial\tau}d\tau$$

$$=\beta(t)x(t)-\beta(t_{-p})x(t_{-p})-\sum_{i=-p}^{0}\int_{t_i}^{t_{i+1}}x(\tau)\beta'(\tau)d\tau$$

$$=\beta(t)x(t)-\beta(t_{-p})x(t_{-p})-$$

$$\sum_{i=-p}^{0}x^m(t_i)[\beta(t_{i+1})-\beta(t_i)] \qquad (4)$$

where $\beta'(\tau)=\partial\beta(\tau)/\partial t$, $t_1=t$, $x^m(t_i)=x(t_m)$, $t_i<t_m<t_{i+1}$

Then a difference-integral equation can be obtained which is called a self-memory equation with the retrospective order p

$$\beta(t)x(t)-\beta(t_{-p})x(t_{-p})-\sum_{i=-p}^{0}x^m(t_i)[\beta(t_{i+1})-\beta(t_i)]$$

$$=\int_{t_{-p}}^{t}\beta(\tau)F(x,\lambda,\tau)d\tau \qquad (5)$$

4. The Richards' equation combined with self-memory principle

The Eq.(1) can be written as

$$\frac{\partial h}{\partial t}=\frac{1}{C(h)}\left[\frac{\partial}{\partial z}\left[K(h)\frac{\partial h}{\partial z}\right]-\frac{\partial K(h)}{\partial z}\right]$$

$$=F(h,z,t) \qquad (6)$$

Defining the memorial function β(i,t) (i as a spatial point, t as time) , and combing Eq.(6) with self-memory principle, the self-memorial Richards' equation can be got

$$\beta(i,t)h(i,t)=\{\beta(i,t_{-p})h(i,t_{-p})$$

$$+\sum_{i=-p}^{0}[\beta(i,t_{i+1})-\beta(i,t_i)]h^m(i,t_i)\}$$

$$+\int_{t_{-p}}^{t}\beta(i,\tau)\frac{1}{C(h(i,\tau))}[\frac{\partial}{\partial z}\left[K(h(i,\tau))\frac{\partial h(i,\tau)}{\partial z}\right]$$

$$-\frac{\partial K(h(i,\tau))}{\partial z}]d\tau \qquad (7)$$

where $h^m(i,t_i)=h(i,t_m)$, $t_i<t_m<t_{i+1}$.

5. The retrospective difference scheme of Richards' equation

The difference scheme of self-memory equation is called the retrospective scheme. Some notations are used in the difference scheme as follows

$$\beta_i^{k+1}=\beta(i,t),\beta_i^k=\beta(i,t_0),\beta_i^{k-1}=\beta(i,t_{-1})\cdots$$

$$h_i^{k+1}=h(i,t),h_i^k=h(i,t_0),h_i^{k-1}=h(i,t_{-1})\cdots$$

$\frac{1}{C(h(i,t))}\left[\frac{\partial}{\partial z}\left[K(h(i,t))\frac{\partial h(i,t)}{\partial z}\right]-\frac{\partial K(h(i,t))}{\partial z}\right]$ can be approximated by

$$\frac{1}{C_i^{k+1}}\left[K_{i+1/2}^{k+1}\frac{h_{i+1}^{k+1}-h_i^{k+1}}{\Delta z^2}-K_{i-1/2}^{k+1}\frac{h_i^{k+1}-h_{i-1}^{k+1}}{\Delta z^2}-\frac{K_{i+1/2}^{k+1}-K_{i-1/2}^{k+1}}{\Delta z}\right]$$

Taking the arithmetic mean values of conductivities at the element interface

$$K_{i\pm1/2}=\frac{1}{2}(K_i+K_{i\pm1})$$

Then the discretization scheme of Eq.(7) can be expressed as follows

$$\beta_i^{k+1}h_i^{k+1}=\beta_i^{k-p}h_i^{k-p}+\sum_{j=-p}^{0}(\beta_i^{k+j+1}-\beta_i^{k+j})\cdot(h_i^{k+j})^m$$

$$+\sum_{j=-p}^{0}\frac{\beta_i^{k+j}}{C_i^{k+j+1}}[K_{i+1/2}^{k+j+1}\frac{h_{i+1}^{k+j+1}-h_i^{k+j+1}}{\Delta z^2}$$

$$-K_{i-1/2}^{k+j+1}\frac{h_i^{k+j+1}-h_{i-1}^{k+j+1}}{\Delta z^2}-\frac{K_{i+1/2}^{k+j+1}-K_{i-1/2}^{k+j+1}}{\Delta z}]\Delta t \qquad (8)$$

Assuming that memorial function β(i,t) of each space point is equal at the same time, that is, β(i,t)= β(i-1,t)=β(i+1,t)…

Then denoting

$$\alpha_{-p-1}=\frac{\beta_i^{k-p}}{\beta_i^{k+1}},\alpha_j=(\beta_i^{k+j+1}-\beta_i^{k+j})\big/\beta_i^{k+1}$$

$$\tilde{\theta}_j=\beta_i^{k+j}\big/\beta_i^{k+1} \qquad (j=-p-1,...,0)$$

$(h_i^{k+j})^m$ is approximated by $1/2(h_i^{k+j+1}+h_i^{k+j})$, then retrospective difference scheme of the Richards' equation can be got

$$h_i^{k+1}=\alpha_{-p-1}h_i^{k-p}+\sum_{j=-p}^{0}\alpha_j\cdot\frac{h_i^{k+j+1}+h_i^{k+j}}{2}$$

$$+\sum_{j=-p}^{0}\frac{\widetilde{\theta}_j}{C_i^{k+j+1}}[K_{i+1/2}^{k+j+1}\frac{h_{i+1}^{k+j+1}-h_i^{k+j+1}}{\Delta z^2}$$

$$-K_{i-1/2}^{k+j+1}\frac{h_i^{k+j+1}-h_{i-1}^{k+j+1}}{\Delta z^2}-\frac{K_{i+1/2}^{k+j+1}-K_{i-1/2}^{k+j+1}}{\Delta z}]\Delta t \qquad (9)$$

Where α_i, θ_i are memory coefficients.

In Eq.(9), setting p=0, $\beta_i^{k+1}=\beta_i^{k+1}=1$, Eq.(9) become

$$h_i^{k+1}=h_i^k+\frac{1}{C_i^{k+1}}[K_{i+1/2}^{k+1}\frac{h_{i+1}^{k+1}-h_i^{k+1}}{\Delta z^2}-K_{i-1/2}^{k+1}\frac{h_i^{k+1}-h_{i-1}^{k+1}}{\Delta z^2}$$

$$-\frac{K_{i+1/2}^{k+1}-K_{i-1/2}^{k+1}}{\Delta z}]\Delta t \qquad (10)$$

Eq.(10) is the implicit difference scheme of the Richards' equation.

Perform algebra to Eq.(9)

$$E_i\,h_{i-1}^{k+1}+F_i\,h_i^{k+1}+G_i\,h_{i+1}^{k+1}=H_i \qquad (11)$$

where $\mu=\dfrac{\Delta t}{\Delta z^2}$

$$E_i=-\widetilde{\theta}_0\frac{K_{i-1/2}^{k+1}}{C_i^{k+1}}\mu$$

$$F_i=1-\frac{\alpha_0}{2}+\mu\cdot\widetilde{\theta}_0\cdot\frac{K_{i-1/2}^{k+1}+K_{i+1/2}^{k+1}}{C_i^{k+1}},$$

$$G_i=-\widetilde{\theta}_0\frac{K_{i+1/2}^{k+1}}{C_i^{k+1}}\mu$$

$$H_i=\alpha_{-p-1}h_i^{k-p}+\sum_{j=-p}^{-1}\alpha_j\cdot\frac{h_i^{k+j+1}+h_i^{k+j}}{2}+\frac{\alpha_0}{2}h_i^k+$$

$$\sum_{j=-p}^{-1}\frac{\widetilde{\theta}_j}{C_i^{k+j+1}}[K_{i+1/2}^{k+j+1}\frac{h_{i+1}^{k+j+1}-h_i^{k+j+1}}{\Delta z^2}-K_{i-1/2}^{k+j+1}\frac{h_i^{k+j+1}-h_{i-1}^{k+j+1}}{\Delta z^2}$$

$$-\frac{K_{i+1/2}^{j+1}-K_{i-1/2}^{j+1}}{\Delta z}]\Delta t-\frac{\widetilde{\theta}_0}{C_i^{k+1}}\Delta t\frac{K_{i+1/2}^{k+1}-K_{i-1/2}^{k+1}}{\Delta z}$$

(i=1,…,n-1)

Then simultaneous the tri-diagonal system can be solved by chase method.

If there is no observed data before the initial time, then we can integrate p steps with implicit scheme. The calculated results can be used as the initial value of the retrospective scheme.

6. Calculation example

The retrospective difference scheme of the Richards' equation is applied to simulate the fixed head vertical infiltration. The experiment data sets was got from Sun(personal communication,2008). The length of the soil cylinder is 2m, inner diameter is 14.5cm. The water level in cylinder can be controlled constant by using connected vessel. After air dry, rolling, and sieving, the soil sample is layered into the cylinder homogeneously according to the designed soil bulk density. The soil sample in this experiment is obtained form the epipedon of the disturbed loess. The soil dry bulk density r_d of this sample is 1.38 (g/cm^2), saturated water content θs is 0.4688 (cm^3/cm^3) [14].

The soil water characteristic curve and soil water diffusivity are fitted by exponential relationship as follows

$$S=10399e^{-12.676\theta}$$

$$D=0.00002e^{29.581\theta}$$

where S is the soil suction (cm), D is unsaturated soil water diffusivity (cm^2/min), θ is the volumetric soil water content (cm^3/cm^3).

Then the unsaturated hydraulic conductivity is obtained

$$K=-D\frac{d\theta}{ds}=1.52\times10^{-10}e^{42.257\theta}$$

where K is the unsaturated hydraulic conductivity (cm/min).

The upper boundary condition is $h(z=0,t)=10399e^{-12.676\theta s}+H$. H is the constant water head (5cm). The lower boundary condition is $h(z=2m,t)=10399e^{-12.676\theta 0}$. The initial volumetric soil water content θ_0 is 0.08 (cm3/cm3). The space interval Δz is 2cm, the time interval Δt is 0.2min, and the given convergence precision is 0.001.

The selection of memory coefficients is important because it influences the stability of the retrospective scheme directly. The coefficients α_i (i=-p-1…0) can be got when θ_i (i=-p…0) were given. So we can generate θ_i randomly in a given range and use Monte-Carlo method to simulate the stable region of θ_i. Setting retrospective order p=1, memory coefficients θ_{-1} and θ_0 are generated randomly between -1 and 1. The coefficient combinations which result calculation convergence are recorded. The infiltration rate's the mean square root error (ε) of each coefficient combination is also calculated. The cumulative infiltration, infiltration rate and ε are calculated by

$$I = A\int_0^z (\theta - \theta_0)dz$$

$$f_i' = \left[I(i) - I(i-1)\right]/(\Delta t \cdot A)$$

$$\varepsilon = \frac{1}{n}\sqrt{\sum_{i=1}^{n}(f_i' - f_i)^2}$$

where I is the cumulative infiltration, A is the cross sectional area of test cylinder, f_i' is the calculated infiltration rate, f_i is the observed infiltration rate.

The stable region of retrospective scheme with 1 order is shown in figure 1. The RMSE of each point in the stable region is shown in figure 2. The ε/θ_0, ε/θ_{-1}scatter diagrams are shown in figure 3 and 4.

The figure 3 shows that the memory coefficient θ_{-1} is more sensitive and has greater effect on the precision.

Setting θ_{-1}=0.124, θ_0=0.554, each infiltration curve for the implicit scheme, retrospective scheme and the observed is shown in figure 5. From figure 5, it can be seen that the retrospective difference scheme can simulate the infiltration better. The RMSE of retrospective scheme is 0.28 (cm^3/min), and that of the implicit scheme is 0.83 (cm^3/min). the figure 6 shows the volumetric soil water content changes with time and depth.

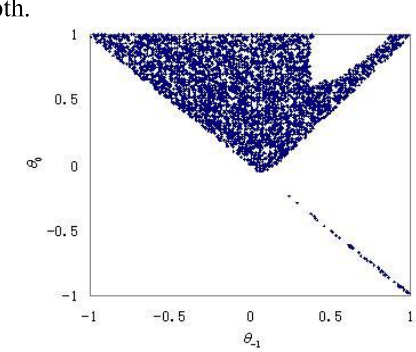

Figure 1. **the stable region**

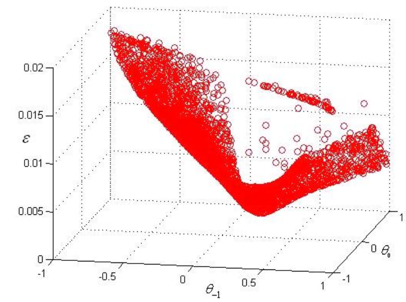

Figure 2. **RMSE of infiltration rate distribution**

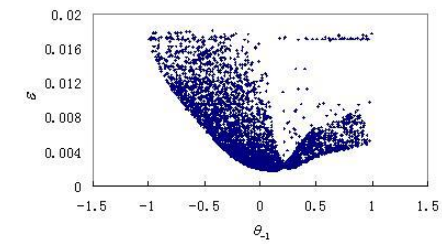

Figure 3. **ε/θ_{-1} scatter diagram**

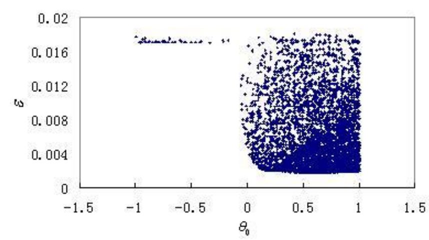

Figure 4. **ε/θ_0 scatter diagram**

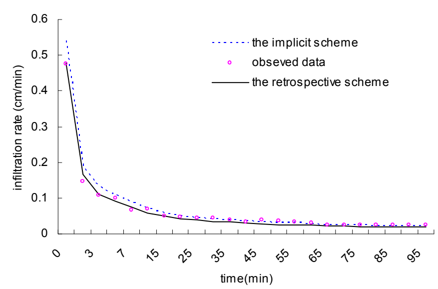

Figure 5. **the infiltration curve of different scheme**

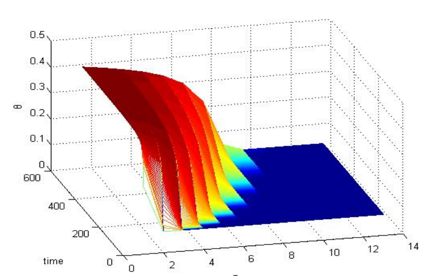

Figure 6. **the change of the volumetric water content**

7. Conclusions

The retrospective difference scheme is flexible because that this scheme has different forms when it was given different memory coefficients. Given some special coefficients can make this scheme have a higher precision. It shows that the retrospective scheme is able to adapt the issue and find the optimal solution for the differential equations. If we can calibrate the memory coefficients with historical observed data, then the scheme will turn into a dynamic-statistic mode which set up a bridge between the deterministic method and the uncertainty method and exert their merits together.

In this paper, the finite difference method was used as the discretization approach of the self-memory equation. We also can combine other numerical method such as the finite element method and the finite volume method with the self-memory equation in the future study.

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